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Fuzzy Logics and Observables

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Systems of fuzzy subsets fulfilling quantum logic axioms with respect to fuzzy connectives are studied. An integral representation of a state on a fuzzy logic is shown. Fuzzy observables and their real-valued mean values are introduced in the obvious way. Using the relationship between fuzzy observables and fuzzy real-valued random variables, a fuzzy real-valued mean value of a fuzzy observable is introduced. The relationship between both types of mean values is studied and an example is presented.

1. INTRODUCTION

One of the most important axiomatic models of quantum mechanics is the quantum logic of Varadarajan (1968). This is a σ-lattice or a σ-poset $\mathbf{L}$ with minimal and maximal element $\mathbf{0}$ and $\mathbf{1}$, respectively, and with a unary operation $\perp : \mathbf{L} \to \mathbf{L}$ ($\perp$ is called an orthocomplementation) so that:

(i) $(a^\perp)^\perp = a$ for any $a \in \mathbf{L}$ (law of repeated negation).
(ii) If $a \leqslant b$, $a, b \in \mathbf{L}$, then $b^\perp \leqslant a^\perp$ (order reversing).
(iii) $a \vee a^\perp = \mathbf{1}$ (excluded middle law) and $a \wedge a^\perp = \mathbf{0}$ (law of contradiction) for any $a \in \mathbf{L}$.
(iv) If $a \leqslant b$, $a, b \in \mathbf{L}$, then there is an element $c \in \mathbf{L}$, $c \leqslant a^\perp$, such that $b = a \vee c$ (orthomodular law).
(v) $\bigvee a_n \in \mathbf{L}$ whenever $\{a_n\} \subset \mathbf{L}$, $a_n \leqslant a_m^\perp$ if $n \neq m$ (σ-orthocompleteness condition).

Two elements a and b are orthogonal and we write $a \perp b$ if $a \leqslant b^\perp$. We recall that a quantum logic is not necessarily distributive or a lattice.

Recently, several fuzzy models of quantum mechanics have appeared. Among these models, only Pykacz's fuzzy quantum logic (Pykacz, 1991) [and generalized fuzzy quantum logic (Pykacz, n.d.)] is a quantum logic

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in the sense of Varadarajan. But then all results for quantum logics can be rewritten for Pykacz's fuzzy quantum logics. Recall that Pykacz used the Giles (1979) fuzzy connectives of bold union and bold intersection for fuzzy sets. In a separate paper (Mesiar, n.d.-a) we have shown that his approach is the only possible one up to an isomorphism. For other fuzzy quantum models, some modifications in the quantum logic axioms are needed. Take, e.g., a fuzzy quantum space M of Riečan (1988). Then axiom (iii) should be weakened to:

(iii*) $\mathbf{m}(a \vee a^{\perp}) = 1$ for any $a \in \mathsf{M}$.

Similarly, (iv) should be weakened to:

(iv*) If $a \leqslant b$, $a, b \in \mathsf{M}$, then there is an element $c \in \mathsf{M}$, $c \leqslant a^{\perp}$, such that $\mathbf{m}(b) = \mathbf{m}(a \vee c)$.

Here **m** is a state (fuzzy probability measure) on M. Moreover, this last model is a distributive one. A similar case is the distributive fuzzy quantum poset of Dvurečenskij and Riečan (1991), which is based on the ideas of Pykacz (1987) [recall that this model is based on the original Zadeh (1965) fuzzy connectives, i.e., $\bigcup = \bigvee$ and $\bigcap = \bigwedge$, $A^{\perp} = 1 - A$).

Since Pykacz's (1991, n.d.) model of a fuzzy quantum logic is rather restrictive from the fuzzy set point of view, we propose another model of fuzzy logic based on Giles fuzzy connectives, where the join and the meet of two elements, respectively, correspond to their bold union and bold intersection, respectively. Consequently, a nonconsistency between the natural ordering of fuzzy sets and the join (meet) may occur.

We define a fuzzy state of a fuzzy logic and we show its relationship to T_{∞}-measures of Klement (1982). Starting from a simple fuzzy observable [corresponding to a simple fuzzy measurable function of Butnariu (1986)], we introduce a fuzzy observable, which corresponds to a fuzzy real-valued random variable of Klement (1985). Finally, we introduce two types of mean value for fuzzy observables—a real-valued one and a fuzzy real-valued one, and we study their relationship.

2. FUZZY LOGIC

Let Ω be a nonvoid set, called the universe. By a fuzzy subset A of Ω we shall understand a real-valued function defined on Ω with values in the unit interval $[0, 1]$, $A: \Omega \to [0, 1]$. The number $A(\omega) \in [0, 1]$ means the degree of membershipness of element ω to the fuzzy set A. The system of all fuzzy subsets of Ω will be denoted by $\mathscr{F}(\Omega)$. The natural ordering on the unit interval induces the ordering on $\mathscr{F}(\Omega)$. We put $A \leqslant B$ iff $A(\omega) \leqslant B(\omega)$ for any $\omega \in \Omega$. The fuzzy connectives of complementation, union, and intersection are defined pointwise, i.e., $A^{\perp}(\omega) = \mathbf{c}(A(\omega))$, $(A \cup B)(\omega) = \mathbf{S}(A(\omega), B(\omega))$, and $(A \cap B)(\omega) = \mathbf{T}(A(\omega), B(\omega))$, $\omega \in \Omega$. Here $\mathbf{c}$ is an order-reversing involution on the unit interval $[0, 1]$. $\mathbf{S}$ and $\mathbf{T}$ form a $\mathbf{c}$-dual pair of a continuous t-conorm and a t-norm, i.e., $\mathbf{T}(x, y) = \mathbf{c}(\mathbf{S}(\mathbf{c}(x), \mathbf{c}(y))$ for any x, y from $[0, 1]$. For more details about fuzzy connectives see, e.g., Dubois and Prade (1985). Recall that the original Zadeh fuzzy complementation is based on the involution $\mathbf{c}(x) = 1 - x$ and hence $A^{\perp} = \mathbf{1} - A$. The Zadeh fuzzy union $\cup$ is based on the t-conorm $\mathbf{S}_0(x, y) = \max(x, y)$ and hence $A \cup B = A \vee B$; similarly, for the Zadeh fuzzy intersection $\cap$ it is $A \cap B = A \wedge B$. In the case of the Giles fuzzy connectives, the bold union $\cup$ is based on the t-conorm $\mathbf{S}_{\infty}(x, y) = \text{in}(x + y, \mathbf{1})$ and hence $A \cup B = \min(A + B, \mathbf{1})$. The bold intersection $\cap$ is based on the t-norm $\mathbf{T}_{\infty}(x, y) = \max(x + y - \mathbf{1}, \mathbf{0})$ and hence $A \cap B = \max(A + B - \mathbf{1}, \mathbf{0})$. Here $\mathbf{0}$ and $\mathbf{1}$ are the minimal and maximal elements of $\mathscr{F}(\Omega)$, respectively.

Let $\mathscr{F}(\Omega)$ be equipped by the fuzzy connectives pointwise generated by $\mathbf{c}$, $\mathbf{S}$, and $\mathbf{T}$. If we take the fuzzy complementation as an orthocomplementation on $\mathscr{F}(\Omega)$, the fuzzy union as the join, and the fuzzy intersection as the meet, then all axioms (i)–(v) are fulfilled if and only if these fuzzy connectives are isomorphic to the Zadeh fuzzy complementation and to the Giles bold union and bold intersection (Mesiar, n.d.-a). This justifies the following definition.

Definition 1. Any nonempty system $\mathscr{L} \subset \mathscr{F}(\Omega)$ of fuzzy subsets of Ω closed under Zadeh fuzzy complementation and under countable Giles bold union will be called a *fuzzy logic* on Ω. The orthocomplementation on $\mathscr{L}$ is the fuzzy complementation $\perp$, the join (meet) is the bold union $\cup$ (bold intersection $\cap$).

Proposition 1. Let $\mathscr{L}$ be a fuzzy logic on Ω. Then $\mathscr{L}$ fulfills the axioms (i)–(v).

Proof. The duality of the Giles bold union and intersection ensures that $\mathscr{L}$ is closed under bold intersections, too. Since $\mathscr{L}$ is nonempty, there is a fuzzy subset $A \in \mathscr{L}$. Then $\mathbf{0} = A \cap A^{\perp} \in \mathscr{L}$ and $\mathbf{1} = \mathbf{0}^{\perp} \in \mathscr{L}$. The validity of axioms (i)–(iii) and (v) is obvious. For (iv), let $A, B \in \mathscr{L}$, $A \leqslant B$. Then $A^{\perp} = \mathbf{1} - A \in \mathscr{L}$, and consequently $C = A^{\perp} \cap B = \max(\mathbf{1} - A + B - \mathbf{1}, \mathbf{0}) = B - A \in \mathscr{L}$. We have $C \leqslant A^{\perp}$ and $A \cup C = \min(A + B - A, \mathbf{1}) = B$. ■

Remark 1. Any fuzzy logic $\mathscr{L}$ is a $\mathbf{T}_{\infty}$-tribe of Klement (1982) [see also Butnariu and Klement (1991)]. Consequently, $\mathscr{L}$ contains the pointwise least upper bound (greatest lower bound) of any sequence of elements of $\mathscr{L}$. Recall that a fuzzy logic need not be a quantum logic in the sense of Varadarajan, as the least upper bound of the sequence of elements of $\mathscr{L}$ may be less than the join of these elements, i.e., than the Giles bold union of these elements. Finally note that the structure of a fuzzy logic is not distributive.

3. FUZZY STATE

A state **m** on a quantum logic **L** maps **L** into the unit interval so that $\mathbf{m}(\mathbf{1}) = 1$ and $\mathbf{m}(\bigvee a_n) = \sum \mathbf{m}(a_n)$ for any sequence $\{a_n\} \subset \mathbf{L}$ of mutually orthogonal elements, i.e., $a_n \perp a_m$ whenever $n \neq m$. The mutual orthogonality of $\{a_n\}$ is equivalent to the total orthogonality of the sequence $\{a_n\}$, which means that for any n, the element a_n is orthogonal to the join of the other elements $\bigvee_{m \neq n} a_m$. The total orthogonality is crucial for defining a state on a quantum logic. On fuzzy logics, the equivalence of the mutual and the total orthogonality may fail. This is why the definition of a fuzzy state of a fuzzy logic should be based on the notion of the total orthogonality.

Definition 2. A fuzzy state **m** on a fuzzy logic $\mathscr{L}$, $\mathbf{m}\colon \mathscr{L} \to [0, 1]$, is a mapping such that $\mathbf{m}(\mathbf{1}) = 1$ and for any totally orthogonal sequence $\{A_n\}$ of elements of $\mathscr{L}$ it is $\mathbf{m}(\bigcup A_n) = \sum \mathbf{m}(A_n)$.

Lemma 1. Let $\mathscr{L}$ be a fuzzy logic. A sequence $\{A_n\} \subset \mathscr{L}$ is totally orthogonal if and only if the algebraic sum $\sum A_n$ is equal to the fuzzy union $\bigcup A_n$.

Proof. Let $\omega \in \Omega$. If $A_n(\omega) = 0$ for any n, then $(\bigcup A_n)(\omega) = 0 = \sum A_n(\omega)$. Let $A_n(\omega) \neq 0$ for some n. The elements A_n and $\bigcup_{m \neq n} A_m$ are orthogonal and hence $\bigcup_{m \neq n} A_m = \min(\sum_{m \neq n} A_m, 1) \leqslant A_n^{\perp} = 1 - A_n$. It follows that $\sum_{m \neq n} A_m(\omega) \leqslant 1 - A_n(\omega)$, and hence $\sum_m A_m(\omega) \leqslant 1$. Consequently, $(\bigcup A_m)(\omega) = \sum A_m(\omega)$, what finishes the proof. ■

Theorem 1. Let $\mathscr{L}$ be a fuzzy logic on Ω. Then **m** is a fuzzy state on $\mathscr{L}$ if and only if there is a probability **P** on the σ-algebra $\mathscr{S}$ of all crisp subsets of Ω contained in $\mathscr{L}$ so that

$$\forall A \in \mathscr{L}\colon \quad \mathbf{m}(A) = \int A \, d\mathbf{P}$$

where the right-hand side is a Lebesgue–Stieltjes integral.

Proof. It is obvious that the system $\mathscr{S}$ of all crisp subsets of Ω contained in $\mathscr{L}$ is a classical σ-algebra. Further, any element A contained in $\mathscr{L}$ is a $\mathscr{S}$-measurable function (Klement, 1982; Butnariu and Klement, 1991). Any fuzzy state $\mathbf{m}$ on $\mathscr{L}$ is a $\mathbf{T}_\infty$-measure of Klement (1982), i.e., a nondecreasing left-continuous valuation satisfying the boundary conditions [it is evident and $\mathbf{m}(\mathbf{0})$ equals $\mathbf{0}$]. Now, the result can be found in Butnariu and Klement (1991). ■

The foregoing theorem shows that any fuzzy state on a fuzzy logic is in fact a fuzzy probability measure introduced by Zadeh (1968).

4. FUZZY OBSERVABLES AND THEIR MEAN VALUES

Let $\mathscr{L}$ be a fuzzy logic on Ω. A fuzzy observable $\mathbf{x}$ of $\mathscr{L}$ is defined in the usual way, i.e., $\mathbf{x}$ is a σ-homomorphism from the system $\mathscr{B}(\mathbf{R})$ of all Borel subsets of the real line into $\mathscr{L}$. Hence a fuzzy observable $\mathbf{x}: \mathscr{L} \to \mathscr{B}(\mathbf{R})$ fulfills:

(α) $\mathbf{x}(E^c) = \mathbf{x}(E)^\perp = \mathbf{1} - \mathbf{x}(E)$ for any $E \in \mathscr{B}(\mathbf{R})$.

(β) $\mathbf{x}(\bigcup E_n) = \bigcup \mathbf{x}(E_n)$ for any sequence $\{E_n\}$ of mutually exclusive subsets from $\mathscr{B}(\mathbf{R})$.

It is evident that $\mathbf{x}(\varnothing) = \mathbf{0}$ and $\mathbf{x}(\mathbf{R}) = \mathbf{1}$. Further, if $\mathscr{L} = \mathscr{F}(\mathscr{S})$ is a generated fuzzy σ-algebra (i.e., the system of all $\mathscr{S}$-measurable fuzzy subsets of Ω), then fuzzy observables of $\mathscr{L}$ coincide with $\mathbf{T}_\infty$-fuzzy observables of Kolesárová and Riečan (1992, n.d.).

Let $\mathbf{m}$ be a fuzzy state on $\mathscr{L}$ and let $\mathbf{x}$ be a fuzzy observable of $\mathscr{L}$. Then $\mathbf{m}_\mathbf{x}(E) = \mathbf{m}(\mathbf{x}(E))$, $E \in \mathscr{B}(\mathbf{R})$, defines a probability distribution on $\mathscr{B}(\mathbf{R})$ (Riečan, 1989). Now we are able to define a crisp mean value of $\mathbf{x}$,

(γ) $\mathbf{M}(\mathbf{x}) = \int_\Omega \mathbf{x}\, d\mathbf{m} = \int_\mathbf{R} t\, d\mathbf{m}_x(t)$

if the right-hand side Lebesgue–Stieltjes integral does exist.

Recently Kolesárová and Riečan (1992, n.d.) showed that $\mathbf{T}_\infty$-fuzzy observables (i.e., fuzzy observables of a generated fuzzy σ-algebra $\mathscr{L}$) are in a one-to-one correspondence with random variables on $(\Omega, \mathscr{S})$ with values in the finite fuzzy real line (Klement, 1985). We have shown (Mesiar, n.d.-b) that the fuzzy real-valued random variables of Klement (1985) are isomorphic to the extended fuzzy observables of $\mathscr{F}(\mathscr{S})$ (the σ-homomorphisms from $\mathscr{B}(\bar{\mathbf{R}})$ into $\mathscr{F}(\mathscr{S})$, where $\bar{\mathbf{R}}$ is the extended real line). Let $\mathbf{x}$ be a fuzzy observable of a fuzzy logic $\mathscr{L}$. Since $\mathscr{L}$ is contained in the generated fuzzy σ-algebra $\mathscr{F}(\mathscr{S})$, $\mathbf{x}$ is a $\mathbf{T}_\infty$-fuzzy observable of $\mathscr{F}(\mathscr{S})$, too. Consequently $\mathbf{x}$ corresponds to a finite fuzzy real-valued random variable $\mathbf{X}$ on $(\Omega, \mathscr{S})$, $\mathbf{X}: \Omega \to \mathscr{F}(\mathbf{R})$. This correspondence is done through:

(δ) $\mathbf{x}(]-\infty, t[)(\omega) = \mathbf{X}(\omega)(t)$, $\omega \in \Omega$, $t \in \mathbf{R}$.

Recall that the system $\mathscr{F}(\mathbf{R})$ of all finite fuzzy reals consists of all distribution functions on $\mathbf{R}$, i.e., $\mathbf{p} \in \mathscr{F}(\mathbf{R})$ iff $\mathbf{p}: \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ is a nondecreasing left continuous mapping such that $\inf \mathbf{p}(t) = 0$ and $\sup \mathbf{p}(t) = 1$.

Remark 2. Let $\mathscr{L} = \mathscr{F}(\mathscr{S})$ be a generated fuzzy σ-algebra. Butnariu (1986) defined a simple fuzzy measurable function $\mathbf{s} = \sum a_i A_i$ as a couple $(\{A_i\}, \{a_i\})$, where $\{A_i\} \subset \mathscr{L}$ is a finite fuzzy partition of $(\Omega, \mathscr{S})$, i.e., the algebraic sum $\sum A_i = \mathbf{1}$, and a_i are real constants. $\mathbf{s}$ induces a σ-homomorphism $\mathbf{s}^{+1}: \mathscr{B}(\mathbf{R}) \to \mathscr{L}$ via:

$$(\varepsilon) \quad \mathbf{s}^{+1}(E) = \sum E(a_i) \cdot A_i, \; E \in \mathscr{B}(\mathbf{R})$$

where $E(\cdot)$ is the characteristic function of the Borel subset E. If all elements A_i are crisp subsets from $\mathscr{S}$, then $\mathbf{s}$ is a real simple function and $\mathbf{s}^{+1}$ is its inverse. For a general simple fuzzy measurable function $\mathbf{s}, \mathbf{s}^{+1}$ may be viewed as a simple fuzzy observable. Using the standard limit procedures, we arrive at the extended fuzzy observables (Mesiar, n.d.-b). Note that $\mathbf{M}(\mathbf{s}^{+1}) = \sum a_i \cdot \mathbf{m}(A_i)$.

Let $\mathbf{m}$ be a fuzzy state on a fuzzy logic $\mathscr{L}$ and let $\mathbf{P}$ be a probability measure generating $\mathbf{m}$ (i.e., $\mathbf{P} = \mathbf{m}/\mathscr{S}$). Let $\mathbf{x}$ be a fuzzy observable of $\mathscr{L}$ and let $\mathbf{X}$ be the corresponding fuzzy real-valued random variable. Klement (1985) defined a mean value of $\mathbf{X}$ with respect to $\mathbf{P}$ by the help of so-called quasi-inverses:

$$(\phi) \quad \mathbf{FM}(\mathbf{X}) = \int_\Omega \mathbf{X} \, d\mathbf{P} = [\int_\Omega \mathbf{X}^{[q]} \, d\mathbf{P}]^{[q]}$$

where the right-hand side (if it exists) is a fuzzy real number (possibly nonfinite). Recall that for a left-continuous nondecreasing mapping $p: [a, b] \to [c, d]$, $p(a) = c$, where a, b, c, and d are some constants from the extended real line, its quasi-inverse $p^{[q]}: [c, d] \to [a, b]$ is defined through $p^{[q]}(c) = a$ and $p^{[q]}(x) = \sup\{y \in [a, b]; \, p(y) < x\}$ for $x \in]c, d]$. It is easy to see that the operation $[q]$ is involutive, i.e., $(p^{[q]})^{[q]} = p$. For more details see Klement (1985) or Kolesárová and Riečan (1992, n.d.).

Definition 3. Let $\mathbf{x}$ be a fuzzy observable of a fuzzy logic $\mathscr{L} \subset \mathscr{F}(\mathscr{S})$. A fuzzy mean value $\mathbf{FM}(\mathbf{x})$ of $\mathbf{x}$ with respect to a given fuzzy state $\mathbf{m}$ on $\mathscr{L}$ (if it exists) is a fuzzy real number $\mathbf{FM}(\mathbf{x}) = \mathbf{FM}(X)$, where $\mathbf{X}$ is the corresponding Klement random variable with values in finite fuzzy real line and $\mathbf{FM}(\mathbf{X})$ is its mean value with respect to $\mathbf{P} = \mathbf{m}/\mathscr{S}$.

The following theorem shows the relationship between the crisp mean value and the fuzzy mean value of a fuzzy observable.

Theorem 2. Let $\mathbf{x}$ be a fuzzy observable of a fuzzy logic $\mathscr{L}$ with finite fuzzy mean value $\mathbf{FM}(\mathbf{x})$ with respect to a fuzzy state $\mathbf{m}$. Then $\mathbf{FM}(\mathbf{x})$ is a probability distribution function on $\mathbf{R}$ and its mean value $\mathbf{M}(\mathbf{FM}(\mathbf{x}))$ (if it exists) equals the crisp mean value $\mathbf{M}(\mathbf{x})$.

Proof. Let $\mathbf{p}$ be a finite fuzzy real number, i.e., $\mathbf{p}$ is a probability distribution function on $\mathbf{R}$. Then the mean value $\mathbf{M}(\mathbf{p})$, if it exists, is a Lebesgue–Stieltjes integral of the quasi-inverse $\mathbf{p}^{[q]}$ over the unit interval with respect to the Lebesgue measure,

$$\mathbf{M}(\mathbf{p}) = \int_{\mathbf{R}} t\, d\mathbf{p}(t) = \int_0^1 \mathbf{p}^{[q]}(\alpha)\, d\alpha$$

Let $\mathbf{FM}(\mathbf{x})$ be a finite fuzzy mean value of $\mathbf{x}$. Let $\mathbf{X}$ be a fuzzy real-valued random variable corresponding to $\mathbf{x}$. Then

$$\begin{aligned}\mathbf{M}(\mathbf{FM}(\mathbf{x})) &= \int_0^1 \left[\int_\Omega (\mathbf{X}(\omega))^{[q]}(\alpha)\, d\mathbf{P}(\omega)\right] d\alpha \\ &= \int_\Omega \left[\int_0^1 (\mathbf{X}(\omega))^{[q]}(\alpha)\, d\alpha\right] d\mathbf{P}(\omega) \\ &= \int_\Omega \mathbf{M}(\mathbf{X}(\omega))\, d\mathbf{P}(\omega) = \int_\Omega \left[\int_{\mathbf{R}} t\, d\mathbf{X}(\omega)(t)\right] d\mathbf{P}(\omega)\end{aligned}$$

On the other hand, the fuzzy state $\mathbf{m}$ induces a probability distribution $\mathbf{m}_{\mathbf{x}}$,

$$\mathbf{m}_{\mathbf{x}}(]-\infty, t[) = \mathbf{m}(\mathbf{x}(]-\infty, t[)) = \int_\Omega \mathbf{x}(]-\infty, t[)(\omega)\, d\mathbf{P}(\omega)$$

for any $t \in \mathbf{R}$. Using equation in condition (δ), we get

$$\begin{aligned}\mathbf{M}(\mathbf{x}) &= \int_{\mathbf{R}} t\, d\mathbf{m}_x(t) = \int_{\mathbf{R}} t\, d\left[\int_\Omega \mathbf{X}(\omega)(t)\, d\mathbf{P}(\omega)\right] \\ &= \int_\Omega \left[\int_{\mathbf{R}} t\, d\mathbf{X}(\omega)(t)\right] d\mathbf{P}(\omega)\end{aligned}$$

The result follows. ■

Example. Let $\Omega = [0, 1]$ and let $\mathscr{L} = \mathscr{F}(\mathscr{B}([0, 1]))$, i.e., the fuzzy logic $\mathscr{L}$ is a generated fuzzy σ-algebra of all Borel-measurable fuzzy subsets of Ω. Let $m(A) = \int_0^1 A\, d\lambda$, $A \in \mathscr{L}$, where λ is the Lebesgue measure. Let $A \in \mathscr{L}$ be a given element. Then $\mathbf{s} = A$ $(= \mathbf{0} \cdot A^\perp + \mathbf{1} \cdot A)$ is a simple fuzzy measurable function (Butnariu, 1986) corresponding to an "indicator function" of A. Then for any $E \in \mathscr{B}(\mathbf{R})$ we have

$$\mathbf{s}^{+1}(E) = \begin{cases} \mathbf{1} \\ A \\ A^{\perp} \\ \mathbf{0} \end{cases} \quad \text{if} \quad E \cap \{0, 1\} = \begin{cases} \{0, 1\} \\ \{1\} \\ \{0\} \\ \varnothing \end{cases}$$

i.e., $\mathbf{s}^{+1}$ is an indicator of the fuzzy subset A introduced by Dvurečenskij and Riečan (1991). Evidently $\mathbf{M}(\mathbf{s}^{+1}) = \mathbf{m}(A)$.

Let $A(\omega) = \omega/2$, $\omega \in [0, 1]$, for example. Then $\mathbf{m}(\mathbf{s}^{+1}) = 1/4$. The corresponding fuzzy real-valued random variable $\mathbf{X}$ is defined as follows:

$$\mathbf{X}(\omega)(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0, \\ 1 - \omega/2 & \text{if} \quad t \in]0, 1], \\ 1 & \text{if} \quad t > 1, \end{cases} \quad \omega \in \Omega, \quad t \in \mathbf{R}$$

We have

$$\mathbf{X}^{[q]}(\omega)(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \leqslant 1 - \omega/2, \\ 1 & \text{if} \quad \alpha > 1 - \omega/2, \end{cases} \quad \omega \in \Omega, \quad \alpha \in [0, 1]$$

Further,

$$\int_{\Omega} (\mathbf{X}^{[q]}(\omega))(\alpha)\, d\lambda(\omega) = \begin{cases} 0 & \text{if} \quad \alpha \leqslant 1/2, \\ 2\alpha - 1 & \text{if} \quad \alpha > 1/2, \end{cases} \quad \alpha \in [0, 1]$$

The fuzzy mean value of the fuzzy observable $\mathbf{s}^{+1}$ is a finite fuzzy real number

$$\mathbf{FM}(\mathbf{s}^{+1})(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0, \\ (1 + t)/2 & \text{if} \quad t \in]0, 1], \\ 1 & \text{if} \quad t > 1, \end{cases} \quad t \in \mathbf{R}$$

It is easy to see that $\mathbf{M}(\mathbf{FM}(\mathbf{s}^{+1})) = \int_0^1 t \cdot (1/2)\, dt = 1/4 = \mathbf{M}(\mathbf{s}^{+1})$. ∎

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